[07-09-09E-T10]

Absolute value inequalities involving rational expressions

■ **[EX1] Find all values of x for which $\left|\frac{x+3}{x-5}\right| > 4$ is true.**

Solution.

Eliminating the absolute value bars results in two cases. They are:

$\frac{x+3}{x-5} > 4$ or $\frac{x+3}{x-5} < -4$

Case 1.

$\frac{x+3}{x-5} > 4$

$\Longleftrightarrow \frac{x+3}{x-5} - 4 > 0$

$\Longleftrightarrow \left(\frac{23-3x}{x-5}\right) > 0$

$\Longleftrightarrow (23 - 3x > 0 \text{ and } x - 5 > 0) \text{ OR } (23 - 3x < 0 \text{ and } x - 5 < 0)$

Hence,

Case [1.1] $23 - 3x > 0$ and $x - 5 > 0$

$23 - 3x > 0 \Longrightarrow x < \frac{23}{3}$ and $x - 5 > 0 \Longrightarrow x > 5$

$x \in (5, \frac{23}{3})$

OR

Case [1.2] $23 - 3x < 0$ and $x - 5 < 0$

$23 - 3x < 0 \Longrightarrow x > \frac{23}{3}$ and $x - 5 < 0 \Longrightarrow x < 5$

$x \in (5, \frac{23}{3}) \bigcap (5, \frac{23}{3})$

$x \in (5, \frac{23}{3})$

So, in Case 1

$x \in (5, \frac{23}{3})$

Case 2.

$\frac{x+3}{x-5} < -4$

$\Longleftrightarrow \frac{x+3}{x-5} + 4 < 0$

$\Longleftrightarrow (\frac{5x-17}{x-5}) < 0$

$\Longleftrightarrow (5x - 17 < 0 \text{ and } x - 5 > 0) \text{ OR } (5x - 17 > 0 \text{ and } x - 5 < 0)$

Hence,

Case [2.1] $5x - 17 < 0$ and $x - 5 > 0$

$5x - 17 < 0 \Longrightarrow x < \frac{17}{5}$ and $x - 5 > 0 \Longrightarrow x > 5$

$x \in (-\infty, \frac{17}{5}) \bigcap (5, \infty +)$

$x \in \phi$

OR

Case [2.2] $5x - 17 > 0$ and $x - 5 < 0$

$5x - 17 > 0 \Longrightarrow x > \frac{17}{5}$ and $x - 5 < 0 \Longrightarrow x < 5$

$x \in (\frac{17}{5}, \infty +) \bigcap (-\infty, 5)$

$x \in (\frac{17}{5}, 5)$

So, in Case 2

$x \in (\frac{17}{5}, 5) \bigcup \phi$

$x \in (\frac{17}{5}, 5)$

Case 1 $\Longrightarrow x \in (5, \frac{23}{3})$ and Case 2 $\Longrightarrow x \in (\frac{17}{5}, 5)$ and either Case 1 or Case 2 is true,

$\therefore x \in (\frac{17}{5}, 5) \bigcup (5, \frac{23}{3})$.